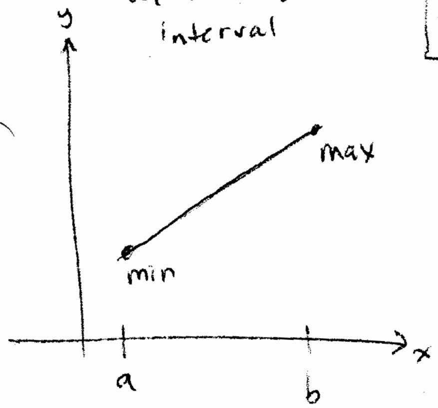
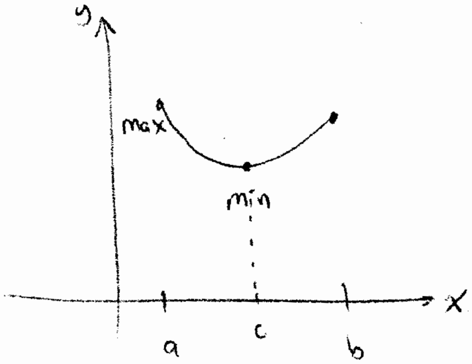
Math 121 3.3 Optimization

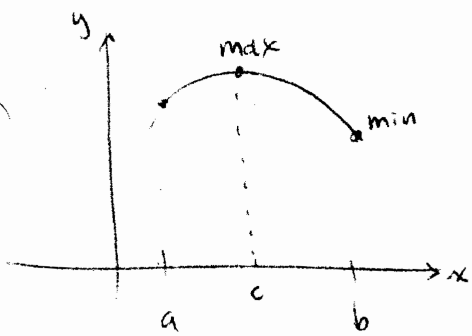
Objectives
1) Understand the difference between an absolute extreme value and a relative extreme value.
2) Understand how constraining a function to a specific interval affects the process of finding absolute extrema.
3) Find absolute extrema on a given interval.
4) Solve applications of optimization. (which we'll continue in 3.4)

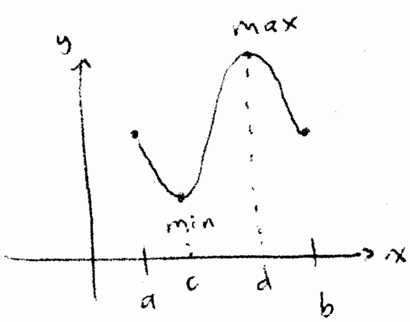
ABSOLUTE EXTREMA

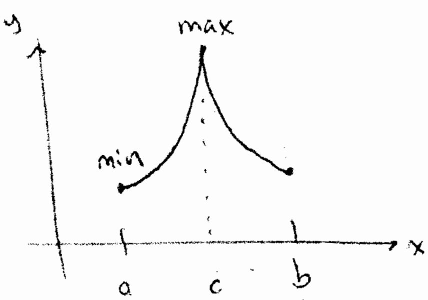
on a closed interval



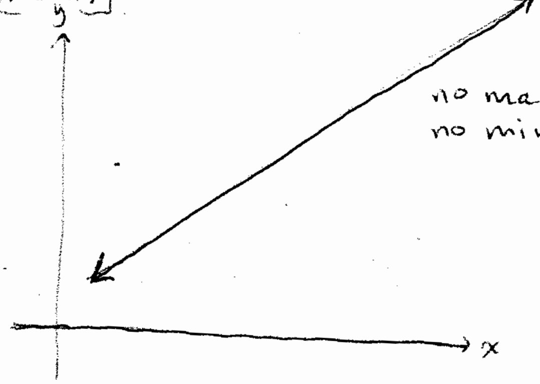


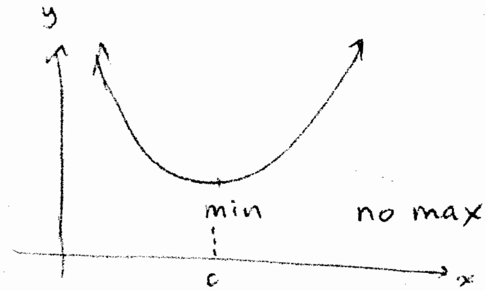


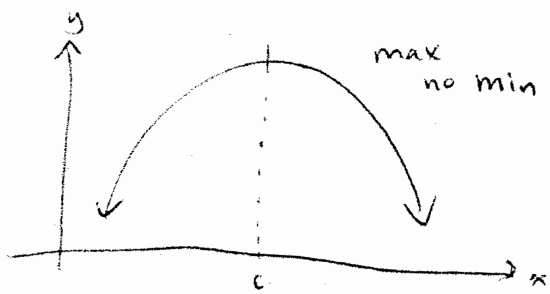


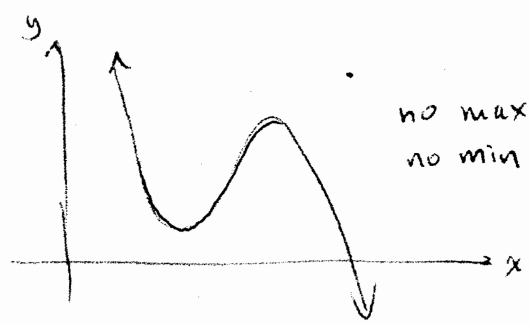


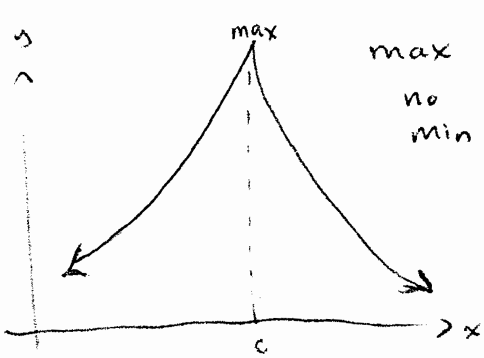
on $(-\infty, \infty)$

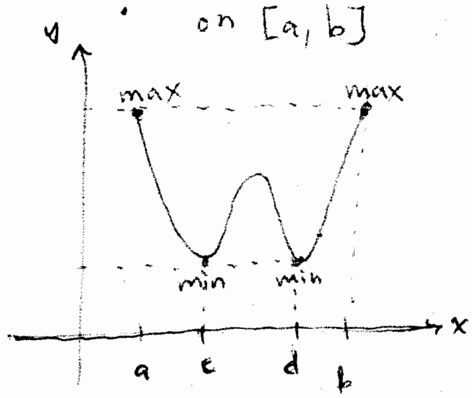


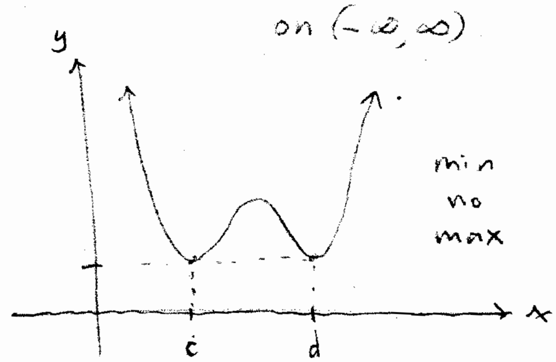












To find absolute extrema on a closed interval, we need to test y-coordinates at

- critical values $f'(x) = 0$
- critical values $f'(x)$ undef
- endpoints

To find absolute extrema on $(-\infty, \infty)$, we have to know the entire graph

However, if the function has only one critical value, concavity at that value can be used to determine if it is an absolute extreme value

$f''(c) > 0$ min

$f''(c) < 0$ max

CAUTION: $f''(c)$ must be defined. So we can't use this for cusps, where $f''(c)$ and $f'(c)$ are undefined.

① Find the absolute extreme values of $f(x) = x^3 - 9x^2 + 15x$ on $[0,3]$.

Notice:

a) The word "absolute" b) the interval given.

These tell us a different process from the one used for relative extrema in 3.2 and 3.1

step 1: Find all critical values. in given interval.

$f'(x) = 3x^2 - 18x + 15$

$f'(x) = 0$ means $\frac{3x^2}{3} - \frac{18x}{3} + \frac{15}{3} = \frac{0}{3}$

$x^2 - 6x + 5 = 0$

5
-5 -1
-6

$(x-5)(x-1) = 0$

$x = 5$ $\quad \underline{\underline{x = 1}}$

↑

not in $[0,3]$

$f'(x)$ undefined? nowhere.

step 2: Make a table of y-coordinates (using the original function) for critical values and endpoints.

x	$f(x)$	
0	0	endpoint
3	-9	endpoint
1	7	critical value

$f(0) = 0^3 - 9(0)^2 + 15(0) = 0$

$f(3) = 3^3 - 9(3)^2 + 15(3) = -9$

$f(1) = 1^3 - 9(1)^2 + 15(1) = 7$

step 3: The largest y-coord is the absolute max. The smallest y-coord is the absolute min.

This function f has:
absolute min -9 (occurring at $x = 3$)
absolute max 7 (occurring at $x = 1$)

You can check this on your calculator for $f(x)$ on $[a,b]$

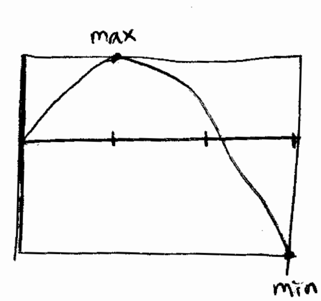
$Y = f(x)$

WINDOW XMIN = a
XMAX = b

YMIN = min you found
YMAX = max you found

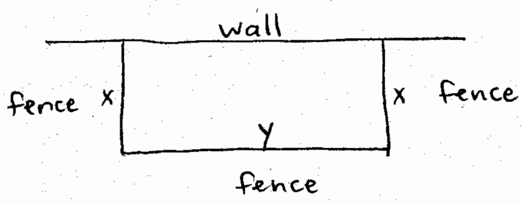
$Y = x^3 - 9x^2 + 15x$

XMIN = 0
XMAX = 3
YMIN = -9
YMAX = 7



② A farmer has 1000 ft of fence and wants to build a rectangular enclosure along a straight wall. If the side along the wall needs no fence, find the dimensions that make the enclosure as large as possible. Then find the maximum area.

<u>step 1</u>: Draw a diagram and label it. Identify variable(s).



<u>step 2</u>: Identify the concept to be maximized or minimized, and its equation. This will be our <u>objective function</u>.

"maximum area" $\Rightarrow$ $A = x \cdot y$

<u>step 3</u>: Identify the given information, and its equation(s). This will be our <u>constraint</u>.

"1000 ft of fence" $\Rightarrow$ $x + x + y = 1000$

$$2x + y = 1000.$$

* Notice: We have 3 variables (A, x, y) and that's too many.

<u>step 4</u>: Using the constraint, use algebra to isolate a variable.

$$2x + y = 1000$$
$$y = -2x + 1000$$

<u>step 5</u>: Substitute this result into the objective function.

} algebra!

$$A = x \cdot y$$
$$A = x(-2x + 1000)$$
$$A = -2x^2 + 1000x$$

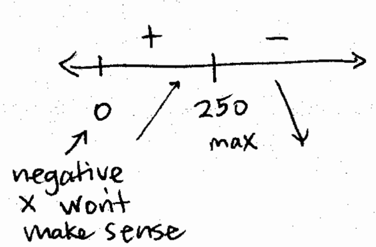
<u>step 6</u>: Find the extrema of constraint
- critical values
- 1st or 2nd derivative test

} calculus!

$A'(x) = -4x + 1000$

$A'(x) = 0$ means $-4x + 1000 = 0$

$$1000 = 4x$$
$$250. = x$$



Step 7: Answer the actual question(s)

"find the dimensions" $\longrightarrow$ we have $x = 250$ ft.
we need y.

$y = -2x + 1000$ (from step 4)

substitute $x = 250$: $y = -2(250) + 1000$

$y = 500$ ft

dimensions are $\boxed{250 \text{ ft} \times 500 \text{ ft}}$.

we used the constraint to answer this question

"find the max area."

$A = x \cdot y$ subst $x = 250$ and $y = 500$

$A = (250)(500)$

$= \boxed{125{,}000 \text{ ft}^2}$

We used the objective function to answer this question.

OR

$A(x) = -2x^2 + 1000x$

$A(250) = -2(250)^2 + 1000(250)$

$= \boxed{125{,}000 \text{ ft}^2}$

* Note: This is the objective function, not its derivative.

③ Find the absolute extrema of $f(x) = \sqrt[3]{x^2}$ on $[-1, 8]$.

$f(x) = x^{2/3}$ denominator of fraction exponent = index of radical

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

$f'(x) = 0$ $2 \neq 0$ no critical values with horizontal tangent lines

$f'(x)$ undefined $3\sqrt[3]{x} = 0$

$\sqrt[3]{x} = 0$

$x = 0$ critical value with vertical tangent line

x	$f(x)$	
-1	1	endpoint
8	4	endpoint
0	0	critical value

$f(-1) = \sqrt[3]{(-1)^2} = \sqrt[3]{1} = 1$

$f(8) = (\sqrt[3]{8})^2 = 2^2 = 4$

$f(0) = \sqrt[3]{0^2} = 0$

absolute minimum 0 occurs at $x = 0$
absolute maximum 4 occurs at $x = 8$

check on GC

$y = x\text{^}(2/3)$

WINDOW XMIN = -1
XMAX = 8
YMIN = 0
YMAX = 4